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A Note on Extension of a Bernoulli Demand Inventory Model

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Abstract
We consider the inventory model with Bernoulli demand pattern that has been discussed in this journal earlier (Sinha, P. (2010). Extension of a Bernoulli Demand Inventory Model. *Operations & Supply Chain Management: An International Journal* 3(1), pp. 30-35). We give a correct analysis for the model, removing the mistakes in that article. We give a search method to obtain an optimal solution exactly. The method is verified in a numerical experiment.

Keywords: *Inventory, Probabilistic, Bernoulli demand.*

1. Introduction

Probabilistic inventory models have an important place in inventory planning literature. Some authors have analyzed such models with demand and/ or supply following Bernoulli distribution or compound Bernoulli distribution. We may mention Johnson (1968), Dunsmuir and Snyder (1989), Janssen et al. (1998), Gullu et al. (1999) as a few expositions of this type. Sinha (2010) [6] discussed a single item (s, Q) (s: reordering position, Q: ordering quantity) type policy with Bernoulli demand and lead time of supply as an integer-valued random variable. A search method was given to find optimal s and Q. However, there are some mistakes in that article so that the analysis and the results need to be corrected.

In this article, we rectify the mistakes in the previous article. We give sufficient conditions so that strictly positive, finite optimal ordering quantity solution exists. Based on such results, a search method is given to determine an optimal solution exactly.

In the next section, we describe the model. In § 3, we give the analysis of the model and the search method. The results of the numerical experiment are also reported here. We conclude with some relevant remarks.

2. Model

We use the same notation as in [6]. The notation and the model are given here for convenience.

Notation

Decision Variables

s : Reordering inventory position, $s = 0, 1, ...;$

Q : Order quantity, $Q = s+1, s+2, ...;$

Parameters

p : Probability of demand being one unit during a unit of time;

L_{av}, L_{min}, L_{max} : Average, minimum and maximum lead time;
c : Cost of shortage per unit;
h : Inventory holding cost per unit, per unit time;
A_0 : Ordering cost for an order;
r : Profit per unit sold (over manufacturing and other such proportional costs);

Other Notation
$\boldsymbol{X}_i$: Random variable denoting demand during the i-th time unit, in a cycle. $\boldsymbol{X}_i = 1$ or 0;
$\boldsymbol{Y}$: Random variable representing lead time of an order. $\boldsymbol{Y} = 0, 1, \ldots$;
$A_1(s, Q)$: Expected inventory carrying cost, depending on s and Q, in a cycle;
$A_2(s)$: Expected shortage cost, depending on s (it does not depend on Q), in a cycle;
$B_0(s, Q)$: Expected time length of a cycle, depending on s and Q.

Model
i) The system starts at time = 0 with an inventory of s units, when an order of replenishment of Q (> s) units is placed.
ii) Demand for the item in every unit of time interval follows a Bernoulli distribution with parameter p. That is, $\boldsymbol{X}_i = 1$ with probability p ($0 < p \le 1$) and $\boldsymbol{X}_i = 0$ with probability $(1 - p)$. The demand occurs at the end of the interval.
iii) Demands at different unit time intervals are mutually independent.
iv) As the inventory position again becomes s, an order of Q units is placed and is received after $\boldsymbol{Y}$ time units, the lead time. $\boldsymbol{Y}$ follows an integer-valued discrete distribution. Minimum value of $\boldsymbol{Y}$ is L_{min} (≥ 0) and maximum value is L_{max} ($< \infty$). Lead times are mutually independent and independent of demands.
v) Demand during a lead time, including the demand, if any, in the last unit time interval in a lead time, is unsatisfied and cost of shortage is c / unit of shortage ($c \ge 0$). Such shortage cost is a measure of goodwill loss, customer dissatisfaction etc.
vi) Inventory holding cost h, ordering cost A_0, profit per unit sold r are non-negative ($h, A_0, r \ge 0$).
vii) Objective function considered is long term cost per unit time.

In the model, since demand during a lead time of supply is lost and $Q \ge s+1$, there would be at most one pending order at any time. There would be no cross-over of orders. At the time of order, inventory position and inventory in hand, i.e., physical stock, both would be s.

3. Analysis

We first give an analysis for the model.

3.1 Analysis

The mistakes in [6] are as:

i. The equation (3.3) in [6] should be as, $V(0, s_1) = (h/(2p))(Q + s_1 - s)(Q + s_1 + s + 1)$.
ii. (3.5) in [6] is wrong. It should be a lower bound as given here in (3.5).
iii. In (3.7) in [6], there should be a – before s/p.
iv. One upper bound for $B_0(s, Q)$ may be as, $B_0(s, Q) \le L_{av} + Q/p$, and not $B_0(s, Q) \le Q/p$.
v. For the above errors, Proposition (3.1) in [6] does not hold.
vi. Numerical experiment gives wrong solutions (although the deviations are not very high).

We give the analysis, correcting the above mistakes, of the model in the succeeding.

(a) Expected Inventory Holding Cost in a Cycle

Let $V(L, s_1)$ denote the average inventory holding cost for the situations when one order is placed at inventory position of s_1, lead time is L and the next order is placed s. We need to find $V(L, s)$. We may note that,

$$V(L, s_1) = h\, s_1 + p\, V(L - 1, s_1 - 1) + (1 - p)\, V(L - 1, s_1), L > 0, s_1 > 0; \quad (3.1)$$

And,

$$V(L,0) = \frac{h}{2p}(Q - s)(Q + s + 1), L \ge 0; \quad (3.2)$$

$$V(0, s_1) = \frac{h}{2p}(Q + s_1 - s)(Q + s_1 + s + 1),\ s_1 > 0. \tag{3.3}$$

The term $V(L, s)$ can be calculated with the above recursive relations. Expected inventory holding cost in a cycle is given as,

$$A_1(s, Q) = \sum_{L=L_{\min}}^{L_{\max}} V(L, s) \Pr\{\boldsymbol{Y}=L\} \tag{3.4}$$

It can be seen that,

$$A_1(s, Q+1) \geq A_1(s, Q) + h(Q+1)/p. \tag{3.5}$$

(b) Expected Shortage Cost in a Cycle

It is given as,

$$A_2(s) = c \sum_{L=s+1}^{L_{\max}} \sum_{k=s+1}^{L} (k - s)\ {}^{L}C_k (1- p)^{L-k} p^k \Pr\{\boldsymbol{Y}=L\} \tag{3.6}$$

A_2 is independent of Q. The term ${}^{L}C_k (1- p)^{L-k} p^k$ may also be calculated recursively.

(c) Expected Time Length of a Cycle

It is obtained as,

$$B_0(s,Q) = \sum_{L=L_{\min}}^{L_{\max}} [L + \frac{Q}{p} - \frac{1}{p} \sum_{k=1}^{\min(L,s)} k\ {}^{L}C_k (1- p)^{L-k} p^k \tag{3.7}$$

$$- \frac{s}{p} \sum_{k=\min(L,s)+1}^{L} {}^{L}C_k p^k (1- p)^{L-k}] \Pr\{\boldsymbol{Y} = L\}$$

with the convention that, in a summation a term is not considered if lower limit is higher than the upper limit. We also get,

$$B_0(s, Q+1) = B_0(s, Q) + 1/p. \tag{3.8}$$

With the application of "Renewal Reward Theorem" (see, for example, Ross 1970), long term cost per unit time (with probability 1) can be written as,

$$K(s, Q) = (- rQ + A_0 + A_1(s, Q) + A_2(s)) / B_0(s, Q). \tag{3.9}$$

We first note a sufficient condition that there is a non-zero optimal Q for the model. Denote, $\overline{Q} = round(\sqrt{2 A_0 p/h})$, i.e., the/a nearest integer to. $\sqrt{2 A_0 p / h}$ We may state the following proposition.

Proposition 3.1: *A sufficient condition such that there is a non-zero optimal Q for the model is that,*

$$r + c > A_0 / \overline{Q} + h(\overline{Q} + 1) / (2p). \tag{3.10}$$

Proof: If we have,

$$cp > \frac{-rQ + A_0 + hQ(Q+1)/(2p) + c\,L_{av}\,p}{(Q/p) + L_{av}}, \tag{3.11}$$

for some $Q = 1, 2, \ldots$, then there is a non-zero optimal Q. This is seen noting that left hand side of (3.11) is the cost per unit time in the long term when $Q = 0$ (the item is not inventoried at all) and right hand side is the same with ($s = 0$, $Q > s$). The condition can be rewritten equivalently as,

$$r + c > A_0 / Q + h(Q + 1)/(2p). \tag{3.12}$$

Right hand side of (3.12) is minimized, without integrality condition, at $Q = \sqrt{2 A_0 p / h}$. From this, we get the condition as stated in the proposition.

With the proposition, an indication of profitability of the item is obtained. In rounding, of the two possible integers which minimizes right hand side (rhs) of (3.12) may also be chosen. But this would not be of much consequence mostly.

We may consider the data as in [6]. These are, $p = 0.1$, $r = 10$/unit, $c = 10$/unit, $h = 0.006$/unit/time, $A_0 = 100$. Then, $\overline{Q} = 58$; $r + c = 20.0$; $A_0 / \overline{Q} = 1.7241$; $h(\overline{Q} + 1)/(2p) = 1.77$. The condition clearly is satisfied.

Consider the next proposition.

Proposition 3.2: *i. For an optimal solution, $s^* \leq L_{max}$; ii. Suppose that, $h > 0$. For any given s, optimal Q is less than equal to the minimum Q such that,*

a. $$\frac{-rQ + A_0 + A_1(s, s+1) + \frac{h}{2p}(Q + s + 2)(Q - s - 1) + A_2(s)}{B_0(s, s+1) + (Q - s - 1)/p} \geq cp \tag{3.12}$$

and

b. $$\frac{h}{2}(Q - s - 1) \geq \frac{-r(s+1) + rp\,B_0(s, s+1) + A_0 + A_1(s, s+1) + A_2(s)}{B_0(s, s+1) + (Q - s - 1)/p} \tag{3.13}$$

Proof: i. For any solution with $s > L_{max}$, we would have another solution with objective function value less or equal, decreasing reordering inventory position to L_{max} and increasing ordering quantity by the same amount.
ii. Let, $h > 0$. We have, $B_0(s, Q) = B_0(s, s+1) + (Q - s - 1)/p$. We may write using (3.5),

$$K(s,Q) \geq \frac{-rQ + A_0 + A_1(s,s+1) + \frac{h}{2p}(Q+s+2)(Q-s-1) + A_2(s)}{B_0(s,s+1) + (Q-s-1)/p} \quad (3.15)$$

The bound in the rhs, and so $K(s, Q)$, increases unboundedly as $Q \rightarrow \infty$. Thus, there must exist a finite optimal Q. Such a solution should have cost less than equal to cp.

Taking the derivative of rhs of (3.15) with respect to Q, the bound given by the rhs is increasing with Q, if the condition in b) is satisfied. If it satisfied for a Q, it is also satisfied for a larger Q.

The following search method now may be suggested. We take that, $h > 0$.

Method
i. Calculate for $s = 0, 1, ..., L_{max}$.
ii. For each s, calculate for $K(s, Q)$ for $Q = s+1, s+2, ...$, until the conditions in (3.13) and (3.14) are satisfied.
iii. Values of s and Q in the search giving minimum $K(s,Q)$ give (s^*, Q^*).

3.2 Numerical Experiment

We re-conduct the numerical experiment as in [6]. Take, $p = 0.1$ (0.2), $r = 10$/unit, $c = 10$/unit (5/unit), $h = 0.006$/unit/time, $A_0 = 100$. One time unit is one hour. Lead time in the first case is constant and is varied as 70, 30, 20, 10, 5, 0 hours. The results are shown in Table 1. In the second case, lead time is distributed in an integer uniform distribution with average as these values except 0, and in each case varies within 20% of the average. The results are in Table 2.

In the instances experimented, optimal ordering quantity is near to what would be given with the economic order quantity (EOQ) formula; optimal reordering point is somewhat more than the average demand during lead time. These would depend on the input data. An optimal decision can be found out without undue difficulty, with the search method as discussed. From the observations of the numerical experiment, the effect of higher lead time often can

Table 1: Optimal Solutions for Constant Lead Time

Obs. No.	Lead Time (hr.)	Optimal Reordering Inventory Position (s^*)	Optimal Ordering Quantity (Q^*)	Cost per Unit Time (hr.) $K(s^*, Q^*)$
$p = 0.1$, $c = 10$				
1	70	9	59	-0.6276
2	30	4	59	-0.6349
3	20	3	59	-0.6381
4	10	2	58	-0.6417
5	5	1	58	-0.6445
6	0	0	58	-0.6506
$p = 0.1$, $c = 5$				
1	70	9	59	-0.6303
2	30	4	59	-0.6373
3	20	3	58	-0.6397
4	10	2	58	-0.6424
5	5	1	58	-0.6453

Table 2: Optimal Solutions for Uniformly Distributed Random Lead Time

Obs. No.	Average Lead Time (hr.)	Optimal Reordering Inventory Position (s^*)	Optimal Ordering Quantity (Q^*)	Cost per Unit Time (hr.) $K(s^*, Q^*)$
$p = 0.1$, $c = 10$				
1	70	9	59	-0.6261
2	30	5	58	-0.6344
3	20	3	59	-0.6378
4	10	2	58	-0.6415
5	5	1	58	-0.6444
$p = 0.1$, $c = 5$				
1	70	9	59	-0.6292
2	30	4	59	-0.6369
3	20	3	58	-0.6395
4	10	2	58	-0.6423
5	5	1	58	-0.6452
$p = 0.2$, $c = 5$				
1	70	18	83	-1.4721
2	30	8	83	-1.4853
3	20	6	82	-1.4895
4	10	3	83	-1.4948
5	5	2	82	-1.4985

be reduced to a large extent in such a model, with a well-designed inventory policy.

The experiment has been done MS Excel, implementing the search method with Visual Basic. A computer with Pentium 4 1.86 GHz processor and Windows XP operating system has been used. The maximum time requirement to solve an instance has been about 1 hr. 25 min. 33 s. This has occurred for the instance with random lead time with average 70.0, $p = 0.2$, $c = 5.0$. The time can be largely reduced if a nearly optimal solution is sought instead of an exactly optimal solution. This may be done varying s in average demand during lead time $\pm$ s', so that the probability that the demand during lead time does not exceed (average demand + s') is 0.99 (service level), etc. The Excel file may be obtained from the author.

4. Concluding Remarks

We have given an exact analysis of a (s, Q) type inventory model with Bernoulli demand, random lead time with an integer-valued distribution and lost sales. The analysis remedies the mistakes in an earlier article. We have discussed a search method to obtain an optimal solution exactly. The method would be adequate for most of the practical instances, particularly because such decision problems are solved off-line. However, a more efficient method may be tried to be found out. The model may also be extended to a multi-item case, with Bernoulli demand but with possible joint replenishment, etc.

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